Fifth and Sixth Integrals for Optimum Rocket Trajectories in a Central Field

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Six integrals of motion are obtained for the fourteenth order problem of minimum fuel rocket trajectories in an inverse square field. These integrals are for both constant exhaust velocity rockets and constant power rockets with unbounded thrust magnitude. The integrals are obtained in two different sets of variables: 1) position and velocity vectors and 2) classical orbit elements. One of the six integrals involves both the payoff and the independent variable, time. The other five integrals involve only the state and adjoint variables. Some applications of the integrals to orbit transfer problems are suggested.

Introduction

THE problem of minimum fuel rocket trajectories in an inverse-square force field involves a set of differential equations of fourteenth order. The Hamiltonian provides a first integral of this problem. Three more integrals are provided by a vector integral due to the spherical symmetry of the problem.[1–6] For constant exhaust velocity rockets with unbounded thrust magnitude two additional integrals have been derived in terms of position and velocity vectors.[3] One of these can be interpreted as the constant Lagrange multiplier for a new payoff variable, the characteristic velocity. The other integral is an integral of the Hamiltonian involving the state, costate, payoff, and time. These two additional integrals have also been derived for the time-open case in terms of classical orbital elements.[5–7]

The present paper derives these two additional integrals for both constant power rockets and constant exhaust velocity rockets, in terms of both position and velocity vectors and classical orbital elements, and for both time open and time fixed cases. The six integrals for the constant power problem are sufficient to theoretically solve some special cases of the general problem.

Derivation—Position and Velocity Vector Formulation

The variational Hamiltonian for minimum fuel rocket trajectories in an inverse square field is given by Eq. (1)[3]

$$H = \frac{\bar{\lambda}\cdot\bar{F}}{m} - \bar{\lambda}\cdot\bar{r}\,\frac{\mu}{r^3} - \dot{\bar{\lambda}}\cdot\dot{\bar{r}} + \sigma\dot{m} \tag{1}$$

The primer vector $\bar{\lambda}$ is the adjoint to the velocity vector $\dot{\bar{r}}$, $\bar{F}$ is the thrust vector, μ is the gravitational constant, and σ is the adjoint to the mass m. The magnitude of the thrust is assumed to be unbounded, Eq. (2)

$$0 \le F \le \infty \tag{2}$$

The equation of motion is given by the second order vector equation, Eq. (3)

$$\ddot{\bar{r}} = \bar{F}/m - \bar{r}(\mu/r^3) \tag{3}$$

The Euler-Lagrange equations are given by the second order vector equation, Eq. (4), and first order scalar equation, Eq. (5)

$$\ddot{\bar{\lambda}} = -\bar{\lambda}(\mu/r^3) + 3\bar{\lambda}\cdot\bar{r}(\mu/r^5)\bar{r} \tag{4}$$

$$\sigma = \bar{\lambda}\cdot\bar{F}/m^2 \tag{5}$$

Two different assumptions will be made for the mass flow rate, Eq. (6). The upper equation corresponds to a rocket with a constant exhaust velocity c. The lower equation corresponds to a low thrust rocket with constant power P.

$$(\dot{m} = -F/c)/(\dot{m} = -F^2/2P) \tag{6}$$

According to the maximum principle, the Hamiltonian, Eq. (1), should be maximized with respect to the direction and magnitude of the thrust. The optimization with respect to direction, is carried out by aligning the thrust with the primer vector $\bar{\lambda}$ to produce Eq. (7)

$$\bar{\lambda}\cdot\bar{F} = \lambda F \tag{7}$$

The optimization with respect to thrust magnitude produces different results for the two types of rockets.

$$\begin{array}{ll} \lambda - \sigma m/c = 0 & F = \infty \\ \lambda - \sigma m/c \equiv 0 & 0 \le F \le \infty \\ \lambda - \sigma m/c < 0 & F = 0 \\ \hline \multicolumn{2}{c}{F = \lambda P/\sigma m} \end{array} \tag{8}$$

For the power-limited rocket, the Hamiltonian has a well-defined maximum given by Eq. (8). For the constant exhaust velocity rocket, the thrust is turned off when the switching function is less than zero. When the switching function is identically zero over a finite arc, singular arcs with finite values of the thrust may arise. Finally, impulses will occur at isolated zeros of the switching function.

It is convenient to introduce two new payoff variables instead of the mass used up to this point. These variables are defined by Eqs. (9)

$$\Delta V \equiv \int_0^t \frac{F}{m}\,dt = c\ln\frac{m}{m^0}\Big/ J = \frac{1}{2}\int_0^t \left(\frac{F}{m}\right)^2 dt \tag{9}$$

The differential equations for the mass multiplier σ may be

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integrated as shown by Eqs. (10)

$$\begin{array}{c} \dfrac{d(\sigma m)}{d\Delta V} = \dfrac{m\dot{\sigma} + \sigma\dot{m}}{F/m} \\ = \lambda - (\sigma m/c) \\ \hline [d(\sigma m^2)/dt] = \dot{\sigma} m^2 + 2\sigma m\dot{m} \\ = 0 \end{array} \tag{10}$$

In the constant exhaust velocity case, the rate of change of σm, with respect to the variable ΔV, will be zero whenever the thrust is turned on, while Eqs. (5) and (6) show that both σ and m will be constant when the thrust is turned off. Therefore, the first integral of the motion will be given by Eq. (11)

$$[\sigma m/c \equiv 1]/[\sigma m^2 \equiv P] \tag{11}$$

The constant values for Eq. (11) are chosen so that the Lagrange multipliers corresponding to the payoff variables of Eq. (9) will be unity. With these choices for the integration constants, Eqs. (12) are obtained.

$$\begin{array}{c} (F/m)(\lambda - 1) = 0 \\ \hline \lambda = F/m \end{array} \tag{12}$$

Note that the primer vector has different physical dimensions for these two problems. For the constant exhaust velocity case the primer vector is dimensionless, while for the constant power rocket the primer vector has the dimensions of acceleration. Incorporating these results into the Hamiltonian, Eq. (1), produces the Hamiltonians for these two problems, Eqs. (13).

$$\begin{array}{c} H = -\bar{\lambda}\cdot\bar{r}(\mu/r^3) - \dot{\bar{\lambda}}\cdot\dot{\bar{r}} \\ \hline H = -\bar{\lambda}\cdot\bar{r}(\mu/r^3) - \dot{\bar{\lambda}}\cdot\dot{\bar{r}} + \lambda^2/2 \end{array} \tag{13}$$

Because the lower equation does not involve time explicitly, the Hamiltonian is a second integral of motion for these problems. The spherical symmetry of the inverse square field results in a vector integral given by Eq. (14).[1–6]

$$\dot{\bar{r}} \times \bar{\lambda} + \dot{\bar{\lambda}} \times \bar{r} = \bar{C} \tag{14}$$

These integrals represent the known integrals for these two problems. In Ref. 3, another integral was obtained for the constant exhaust velocity case. This integral will now be rederived along with the corresponding result for the power-limited rocket. The dot product of the primer vector with Eq. (3) produces Eq. (15)

$$\bar{\lambda}\cdot\ddot{\bar{r}} = \lambda(F/m) - \bar{\lambda}\cdot\bar{r}(\mu/r^3) \tag{15}$$

The dot product of the position vector with Eq. (4) produces Eq. (16)

$$\bar{r}\cdot\ddot{\bar{\lambda}} = 2\bar{\lambda}\cdot\bar{r}(\mu/r^3) \tag{16}$$

Adding twice Eq. (16) to Eq. (15) produces Eq. (17).

$$2\cdot\bar{r}\ddot{\bar{\lambda}} + \bar{\lambda}\cdot\ddot{\bar{r}} = 3\bar{\lambda}\cdot\bar{r}(\mu/r^3) + \lambda(F/m) \tag{17}$$

The time derivative of the left-hand side of Eq. (18) is equal to its right-hand side. Subsituting Eq. (17) into the right-hand side produces Eq. (19)

$$(d/dt)(\bar{\lambda}\cdot\dot{\bar{r}} + 2\bar{r}\cdot\dot{\bar{\lambda}}) = 2\bar{r}\cdot\ddot{\bar{\lambda}} + \bar{\lambda}\cdot\ddot{\bar{r}} + 3\dot{\bar{\lambda}}\cdot\dot{\bar{r}} \tag{18}$$

$$\frac{d}{dt}(\bar{\lambda}\cdot\dot{\bar{r}} + 2\bar{r}\cdot\dot{\bar{\lambda}}) = 3\bar{\lambda}\cdot\bar{r}\left(\frac{\mu}{r^3}\right) + 3\dot{\bar{\lambda}}\cdot\dot{\bar{r}} + \lambda\left(\frac{F}{m}\right) \tag{19}$$

Substituting Eqs. (13) into Eq. (19) produces Eqs. (20).

$$\begin{array}{c} d/dt(\bar{\lambda}\cdot\dot{\bar{r}} + 2\bar{r}\cdot\dot{\bar{\lambda}}) = \lambda(F/m) - 3H \\ \hline d/dt(\bar{\lambda}\cdot\dot{\bar{r}} + 2\bar{r}\cdot\dot{\bar{\lambda}}) = 5(\lambda^2/2) - 3H \end{array} \tag{20}$$

Equations (20) may be integrated with respect to time to produce the sixth integral of motion for these two problems, Eqs. (21)

$$\begin{array}{c} 3Ht + \bar{\lambda}\cdot\dot{\bar{r}} + 2\bar{r}\cdot\dot{\bar{\lambda}} - \bar{\lambda}^0\cdot\dot{\bar{r}}^0 - 2\bar{r}^0\cdot\dot{\bar{\lambda}}^0 = \Delta V \\ \hline 3Ht + \bar{\lambda}\cdot\dot{\bar{r}} + 2\bar{r}\cdot\dot{\bar{\lambda}} - \bar{\lambda}^0\cdot\dot{\bar{r}}^0 - 2\bar{r}^0\cdot\dot{\bar{\lambda}}^0 = 5J \end{array} \tag{21}$$

Orbital Element Formulation

In recent years, much of the progress in space trajectory optimization has come about by using orbital elements as state variables in the variational formulation.[5–13] In this section, the equations and results of the last section will be transformed into the corresponding results in the orbital element formulation. The vector equation of motion, Eq. (3), is replaced by the following six first-order equations for the rates of change of the orbit elements. The elements chosen are semimajor axis a, eccentricity e, argument of perigee ω, inclination i, longitude of the node Ω, and mean anomaly M. Results for other sets of elements are easily derived by the same procedure. The rates of change of these elements may be found in any standard text on celestial mechanics and are given by Eqs. (22–27).

$$\frac{da}{dt} = 2\left(\frac{a^3}{\mu}\right)^{1/2} \frac{F_r e f_1 + F_\theta f_5}{m f_3} \tag{22}$$

$$\frac{de}{dt} = \left(\frac{a}{\mu}\right)^{1/2} \frac{F_r f_1 f_5 + F_\theta(f_4 + f_2 f_3)}{m f_3} - f_5 \tag{23}$$

$$\frac{d\omega}{dt} = \left(\frac{a}{\mu}\right)^{1/2} \frac{-F_r f_4 f_5 + F_\theta(f_3 + f_5^2) f_1}{m e f_3} - \left(\frac{a}{\mu}\right)^{1/2} \frac{F_z(f_1 f_5 \cos\omega + f_4 \sin\omega)}{m f_5 \tan i} \tag{24}$$

$$\frac{di}{dt} = \left(\frac{a}{\mu}\right)^{1/2} \frac{F_z(f_4 \cos\omega - f_1 f_5 \sin\omega)}{m f_5} \tag{25}$$

$$\frac{d\Omega}{dt} = \left(\frac{a}{\mu}\right)^{1/2} \frac{F_z(f_1 f_5 \cos\omega + f_4 \sin\omega)}{m f_5 \sin i} \tag{26}$$

$$\frac{dM}{dt} = \left(\frac{\mu}{a^3}\right)^{1/2} + \left(\frac{a}{\mu}\right)^{1/2} \frac{F_r(f_4 f_5^2 - 2e f_3^2) - F_\theta f_1 f_5 (f_3 + f_5^2)}{m e f_3} \tag{27}$$

The mean and eccentric anomalies are related by Kepler's Eq. (28) and the f_i are given by Eq. (29).

$$M = E - e \sin E \tag{28}$$

$$\begin{array}{c} f_1 \equiv \sin E,\ f_3 \equiv 1 - e\cos E,\ f_2 \equiv \cos E \\ f_4 \equiv \cos E - e,\ f_5 \equiv (1 - e^2)^{1/2} \end{array} \tag{29}$$

The Hamiltonian for this formulation is given by Eq. (30)

$$H = \lambda_a \dot{a} + \lambda_e \dot{e} + \lambda_\omega \dot{\omega} + \lambda_i \dot{i} + \lambda_\Omega \dot{\Omega} + \lambda_M \dot{M} + \sigma \dot{m} \tag{30}$$

The canonical transformation between the two sets of variables is a point transformation that does not involve the time. The new state variables depend only upon the old state variables. Because time is not involved, the Hamiltonian of Eq. (30) is the same as the Hamiltonian of Eq. (1). The primer vector and its derivative may be evaluated by noting that they are independent of the thrust level so that $\bar{\lambda}$ and $\dot{\bar{\lambda}}$ have the same value on the osculating unpowered conic and on the powered trajectory. For the osculating conic the thrust will be zero which yields Eq. (31) when Eq. (1) and (30) are equated

$$-\bar{\lambda}\cdot\bar{r}(\mu/r^3) - \dot{\bar{\lambda}}\cdot\dot{\bar{r}} = \lambda_M(\mu/a^3)^{1/2} \tag{31}$$

The components of the primer vector in the cylindrical coordinate systems whose axis is perpendicular to the orbital plane may be determined by equating the components of the primer vector in Eq. (1) and (30) taking note of Eq. (7).

$$\lambda_r = \left(\frac{a}{\mu}\right)^{1/2} \frac{2\lambda_a ae^2 f_1 + \lambda_e e f_1 f_5{}^2 - (\lambda_\omega - f_5\lambda_M) f_4 f_5 - 2\lambda_M e f_3{}^2}{e f_3} \tag{32}$$

$$\lambda_\theta = \left(\frac{a}{\mu}\right)^{1/2} \times \frac{2\lambda_a aef_5 + \lambda_e ef_5(f_4 + f_2 f_3) + (\lambda_\omega - f_5\lambda_M)(f_3 + f_5{}^2)f_1}{ef_3} \tag{33}$$

$$\lambda_z = \left(\frac{a}{\mu}\right)^{1/2} \frac{f_4\beta_1 + f_1 f_5 \beta_2}{f_5} \tag{34}$$

The quantities β_1 and β_2 are given by Eqs. (35). These definitions are taken from Ref. 6 which will be utilized in the subsequent development.

$$\begin{aligned}\beta_1 &\equiv \lambda_i \cos\omega + \lambda_\Omega \sin\omega \csc i - \lambda_\omega \sin\omega \cot i \\ \beta_2 &\equiv -\lambda_i \sin\omega + \lambda_\Omega \cos\omega \csc i - \lambda_\omega \cos\omega \cot i\end{aligned} \tag{35}$$

The derivative of the primer vector is calculated by determining its value on the osculating unpowered ellipse. The Hamiltonian on this ellipse is given by Eq. (36)

$$H = \lambda_M(\mu/a^3)^{1/2} \tag{36}$$

With this Hamiltonian, the rates of change of the Lagrange multipliers for all the elements except the semimajor axis are zero. The rate of change of the Lagrange multiplier for the semimajor axis is given by Eq. (37).

$$\dot{\lambda}_a = \tfrac{3}{2}\lambda_M(\mu/a^5)^{1/2} \tag{37}$$

The components of the derivative of the primer vector in the r,θ,z directions are given by Eqs. (38–40)

$$\dot{\lambda}_r = \frac{\partial\lambda_r}{\partial E}\dot{E} + \frac{\partial\lambda_r}{\partial\lambda_a}\dot{\lambda}_a - \lambda_\theta\dot{\theta} \tag{38}$$

$$\dot{\lambda}_\theta = \frac{\partial\lambda_\theta}{\partial E}\dot{E} + \frac{\partial\lambda_\theta}{\partial\lambda_a}\dot{\lambda}_a + \lambda_r\dot{\theta} \tag{39}$$

$$\dot{\lambda}_z = \frac{\partial\lambda_z}{\partial E}\dot{E} + \frac{\partial\lambda_z}{\partial\lambda_a}\dot{\lambda}_a \tag{40}$$

The first two terms on the right-hand side represent the rates of change due to the variables in the equations while the last terms on the right-hand side are due to the rotation of the coordinate system.[14] The rates of change of the eccentric anomaly and central angle are given by standard two-body formulas.

$$\dot{E} = \left(\frac{\mu}{a^3}\right)^{1/2}\frac{1}{f_3},\ \dot{\theta} = \left(\frac{\mu}{a^3}\right)^{1/2}\frac{f_5}{f_3{}^2}$$

Carrying out these operations results in Eqs. (41–43) for the components of the rates of change of the primer vector

$$\dot{\lambda}_r = \frac{-2\lambda_a ae - \lambda_e ef_2 f_5 - (\lambda_\omega - f_5\lambda_M)f_1 f_5 + \lambda_M e^2 f_1 f_3}{aef_3{}^2} \tag{41}$$

$$\dot{\lambda}_\theta = \frac{-\lambda_e ef_1 f_5 + (\lambda_\omega - f_5\lambda_M)f_2 + \lambda_M ef_5}{aef_3} \tag{42}$$

$$\dot{\lambda}_z = \frac{-\beta_1 f_1 f_5 + \beta_2 f_2}{af_3 f_5} \tag{43}$$

These six expressions for the primer vector and its rates of change are general expressions which are true both for powered and unpowered orbits. They represent the results of a canonical transformation between the two forms of the equations of motion. In the particular case of an unpowered coasting arc, the osculating orbit will be identical with the coasting arc at all points and Eq. (37) may be integrated to yield Eq. (44).

$$\lambda_a = \lambda_a{}^0 + \tfrac{3}{2}\lambda_M(\mu/a^5)^{1/2}(t - t^0) \tag{44}$$

On a coasting arc, the Lagrange multipliers for all elements except the semimajor axis will be constant. The results obtained herein agree with those obtained by more direct methods in Ref. 14. It should also be noted that the equations for the primer vector and its derivative on a coasting arc are the same as the variational equations for position and velocity.[3] This means that any of the innumerable solutions of the variational equations of two-body orbits may be used to determine the primer vector time history during coast. The present formulation has the possible advantage of identifying the integration constants in terms of the Lagrange multipliers of the orbital elements.

The constants of the motion derived in the last section will now be expressed in terms of the orbital elements and their multipliers. This will be done by direct calculation of the dot and cross products of the primer vector with the position and velocity vectors. The components of position and velocity are given by the following standard two-body equations

$$r = af_3 \quad \dot{r} = \left(\frac{\mu}{a}\right)^{1/2}\frac{ef_1}{f_3} \quad r\dot{\theta} = \left(\frac{\mu}{a}\right)^{1/2}\frac{f_5}{f_3}$$

The first constant of the motion, the Hamiltonian, is given by Eq. (45) for the constant exhaust velocity problem and for the constant power problem

$$\frac{H = (\mu/a^3)^{1/2}\lambda_M}{H = \dot{\bar{\lambda}}\cdot\dot{\bar{\lambda}}/2 + (\mu/a^3)^{1/2}\lambda_M} \tag{45}$$

It should be noted that for the constant exhaust velocity problem, this provides a determination of λ_M in terms of the current state. The value of the primer vector for the power limited problem may be determined by squaring and adding Eqs. (32–34).

The vector constant will be determined by direct calculation to be given by Eq. (46) where the components of the column vector on the right side are in the r,θ, and z directions

$$\dot{\bar{r}} \times \bar{\lambda} + \dot{\bar{\lambda}} \times \bar{r} = \begin{pmatrix} \dfrac{f_4\beta_1 + f_1 f_5\beta_2}{f_3} \\ \dfrac{f_1 f_5\beta_1 - f_4\beta_2}{f_3} \\ \lambda_\omega \end{pmatrix} \tag{46}$$

This result is precisely the result obtained in Ref. 6 for the time open constant exhaust velocity problem. It is shown in that paper that Eq. (46) yields the following equations for the Lagrange multipliers of the Euler angles of the orbit

$$\lambda_\Omega = \lambda_\Omega{}^0 \tag{47}$$

$$\lambda_\omega = \lambda_\Omega{}^0 \cos i + \sin i(C_1 \sin\Omega - C_2\cos\Omega) \tag{48}$$

$$\lambda_i = C_2 \sin\Omega + C_1\cos\Omega \tag{49}$$

The two constants in these equations are given by Eqs. (50) and 51)

$$C_1 = \lambda_i{}^0 \cos\Omega^0 - \lambda_\Omega{}^0 \cot i^0 \sin\Omega^0 + \lambda_\omega{}^0 \sin\Omega^0 \csc i^0 \tag{50}$$

$$C_2 = \lambda_i{}^0 \sin\Omega^0 + \lambda_\Omega{}^0 \cot i^0 \cos\Omega^0 - \lambda_\omega{}^0 \cos\Omega^0 \csc i^0 \tag{51}$$

It should be noted that Eqs. (47–49) are simply the dot products of the vector constant with the axes of rotation of the Euler angles.

Finally, the quantity in the sixth integral may be determined by direct calculation to be given by Eq. (52)

$$\bar{\lambda}\cdot\dot{\bar{r}} + 2\bar{r}\cdot\dot{\bar{\lambda}} = -2\lambda_a a \tag{52}$$

Eqs. (21) then provide a direct representation of the Lagrangian multiplier for the semimajor axis in terms of the Hamiltonian, the time, the cost, and the semimajor axis. It should be noted that Eq. (21) reduces to Eq. (44) on a coasting arc. It should also be noted that if the log of the scalar angular momentum is used as a variable instead of the semimajor axis, then Eq. (52) becomes Eq. (53)

$$\bar{\lambda}\cdot\dot{\bar{r}} + 2\bar{r}\cdot\dot{\bar{\lambda}} = -\lambda_L \tag{53}$$

$$L \equiv \ln h \tag{54}$$

This reproduces a result derived in Refs. 5–7.

Applications

For constant exhaust velocity rockets, Eq. (45) yields a value for λ_M, Eq. (47) for λ_Ω, Eq. (48) for λ_ω, Eq. (49) for λ_i, and Eqs. (52) and (21) yield λ_a. The remaining Lagrange multiplier λ_e may be obtained from the condition that the magnitude of the primer vector must be unity during thrusting arcs. During coasting arcs, λ_e will be constant. The difficult problem in solving impulsive transfers is determining the times at which the impulses occur. Because of this problem, the integrals obtained herein may not be very helpful in solving for optimum impulsive trajectories. Where these integrals may be useful is in the treatment of singular arcs. The existing literature on singular arcs[14,15] has generally been formulated in position and velocity coordinates. The orbit element formulation might yield new insights on this problem.

For constant power rockets, Eq. (37) yields a value for λ_Ω, Eq. (48) for λ_ω, Eq. (49) for λ_i, and Eqs. (52) and (21) yield λ_a. Unlike the constant exhaust velocity case, Eq. (45) does not directly determine λ_M because λ_e also appears in this equation. If one more integral could be found, all of the Lagrange multipliers could be expressed in terms of the state, the payoff, and the time; and the problem could theoretically be reduced to quadratures.[16]

Two constant power problems can be reduced to quadratures by use of these integrals. The first problem is power-limited orbit transfer in a strong gravity field. This problem can be treated by averaging the differential equations.[12] For the averaged problem the short period terms are neglected and only secular terms are considered. For the averaged orbit transfer problem, the averaged value of λ_M will be identically 0, and Eq. (45) then determines λ_e. The semimajor axis is separable from the other coordinates and may be determined simply. The remaining fourth-order system of equations can be reduced to quadratures.

The other constant power problem which can be solved by quadrature, is that of purely radial motion. For this problem, λ_e will be identically 0, and Eq. (45) would then yield λ_M. Alternately, position and velocity coordinates may be used to solve this problem. For example, Eqs. (13) and (21) may be used to solve for r and $\dot{r}$ in terms of the primer vector, its derivatives, and the payoff. This separates the adjoint equations from the equations of motion. Alternately, this problem can theoretically be solved by quadratures by using the techniques in Ref. 16.

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